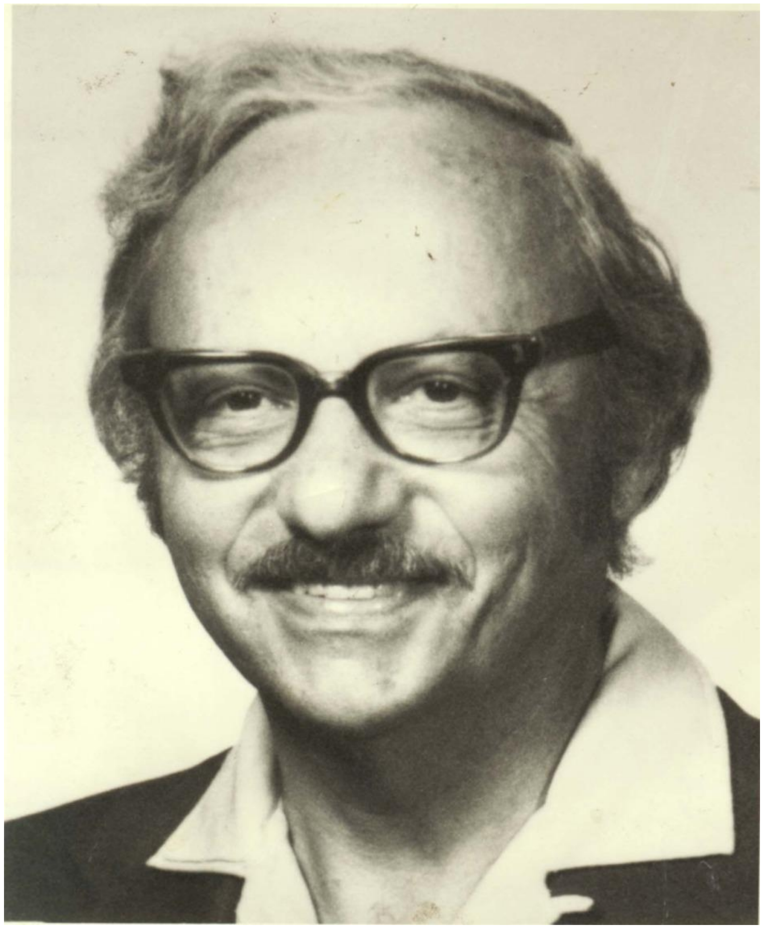
MDI210 : Linear Programming

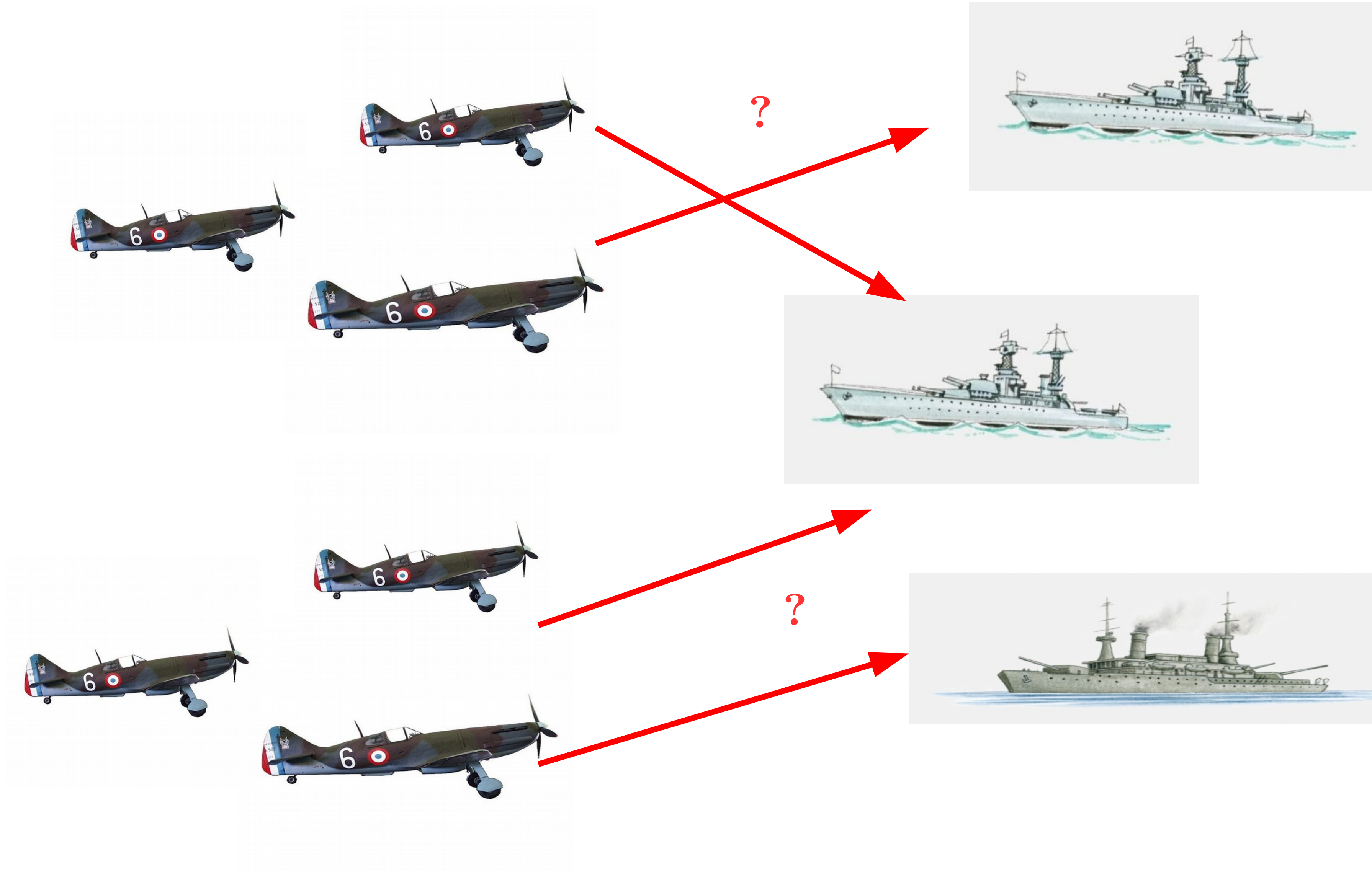
Robert M. Gower

Linear Programming History (1939)

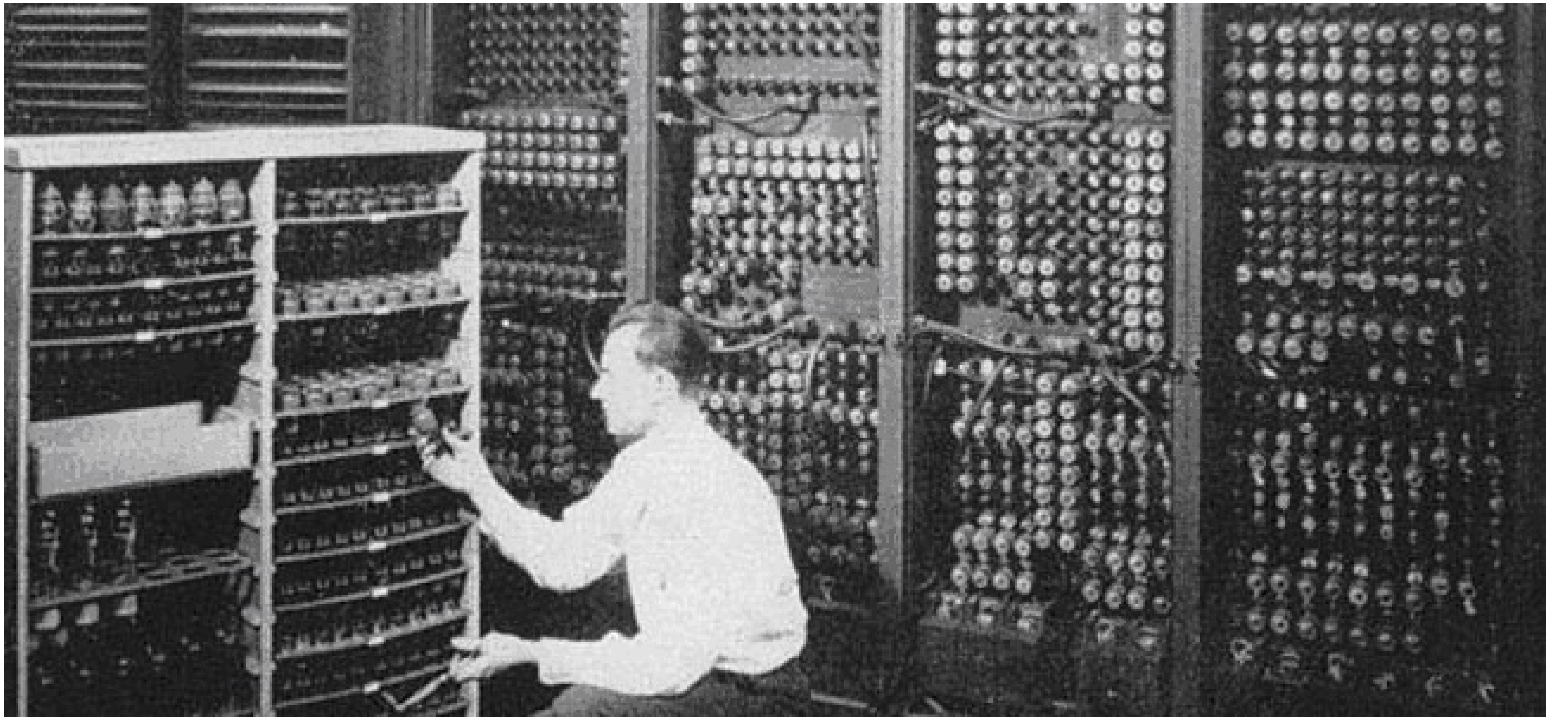


- 1947: George Dantzig, advising U.S. Air Force, invents Simplex.
- Assignment 70 people to 70 jobs (more possibilities than particles).

Linear Programming History (1941)



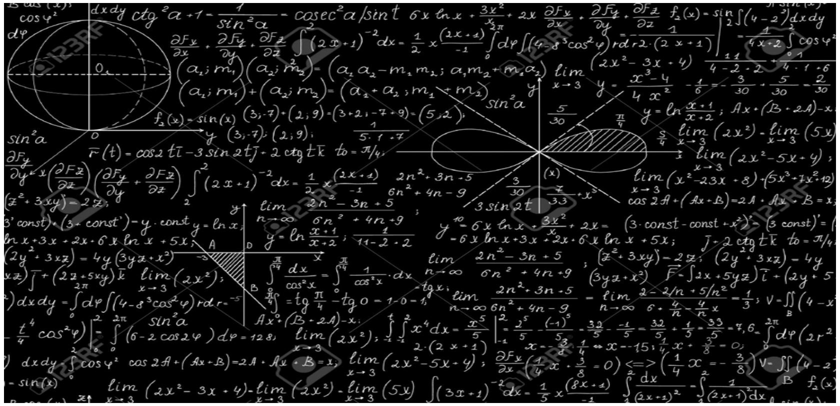
Army Builds Killing Machine (1949)



1949 SCOOP: Scientific Computation Of Optimal Programs

Mathematical Programming: Math used to figured out Flight and logistic programs/schedules

Dantzig the Urban Legend



Dantzig, George B. "On the Non-Existence of Tests of 'Student's' Hypothesis Having Power Functions Independent of Sigma." Annals of Mathematical Statistics. No. 11; 1940 (pp. 186-192).

Dantzig, George B. and Abraham Wald. "On the Fundamental Lemma of Neyman and Pearson." Annals of Mathematical Statistics. No. 22; 1951 (pp. 87-93).

Optimization and Numerical Analysis: Linear Programming

Robert Gower

September 19, 2019

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- Finding an initial feasible dictionary

Duality

The Problem: Linear Programming

$$\max_x z \stackrel{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \leq b,$$
$$x \geq 0,$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. Equivalently

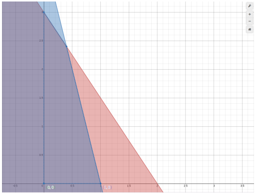
$$\max_x z \stackrel{\text{def}}{=} \sum_{j=1}^{n} c_j x_j$$
$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad \text{for } i = 1, \ldots, m.$$
$$x \geq 0.$$

First example Simplex

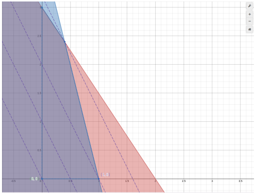
The problem

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 & \\ & 3x_1 + 2x_2 & \leq 600 \\ & 4x_1 + 1x_2 & \leq 400 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

We can solve this graphically:



With level sets $\Rightarrow$



How to do this systematically?

First example Simplex

The problem

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 & \\ & 3x_1 + 2x_2 & \leq 600 \\ & 4x_1 + 1x_2 & \leq 400 \\ & x_1 \geq 0, x_2 \geq 0. & \end{aligned}$$

Can be transformed into

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 \\ x_3 = 600 & - 3x_1 - 2x_2 \\ x_4 = 400 & - 4x_1 - x_2, \end{aligned}$$

where x_3 and x_4 are *slack variables*. This is known as the the *dictionary* format and is often written as:

$$\begin{array}{ccccccc} x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\ x_4 & = & 400 & - & 4x_1 & - & x_2 \\ \hline z & = & & & 4x_1 & + & 2x_2 \end{array}$$

First example Simplex

The *dictionary* format

$$\begin{array}{ccccccc} x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\ x_4 & = & 400 & - & 4x_1 & - & x_2 \\ \hline z & = & & & 4x_1 & + & 2x_2 \end{array}$$

admits obvious solution

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 600, 400).$$

The objective z will improve if $x_1 > 0$. Increasing x_1 as much as possible

$$\begin{array}{ccccc} x_3 \geq 0 & \Rightarrow & 600 - 3x_1 \geq 0 & \Rightarrow & x_1 \leq 200, \\ x_4 \geq 0 & \Rightarrow & 400 - 4x_1 \geq 0 & \Rightarrow & x_1 \leq 100. \end{array}$$

Thus $x_1 \leq 100$ to guarantee $x_4 \geq 0$. This means x_4 will leave the basis and x_1 will enter the basis. Using row operations $z \leftarrow z + r_2$ and $r_1 \leftarrow r_1 - \frac{3}{4} r_2$ to isolate x_1 on row$_2$.

$$\begin{array}{ccccccc} x_3 & = & 300 & + & \frac{3}{4}x_4 & - & \frac{5}{4}x_2 \\ x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\ \hline z & = & 400 & - & x_4 & + & x_2 \end{array}$$

First example Simplex

From

$$\begin{array}{ccccccc} x_3 & = & 300 & + & \frac{3}{4}x_4 & - & \frac{5}{4}x_2 \\ x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\ \hline z & = & 400 & - & x_4 & + & x_2 \end{array}$$

Now we are at the vertex $(x_1^*, x_2^*) = (100, 0)$. Next we see that increasing x_2 increases the objective value but

$$\begin{array}{lllll} x_3 \geq 0 & \Rightarrow & 300 - \frac{5}{4}x_2 \geq 0 & \Rightarrow & 240 \geq x_2, \\ x_1 \geq 0 & \Rightarrow & 100 - \frac{x_2}{4} \geq 0 & \Rightarrow & 400 \geq x_2. \end{array}$$

Increase x_2 upto 240 while respecting the positivity constraints of x_3. Thus x_3 will *leave* the basis and x_2 will *enter* the basis. Performing a row elimination again via $z \leftarrow z + \frac{4}{5}r_1$ and $r_2 \leftarrow r_2 - \frac{1}{5}r_1$, we have that

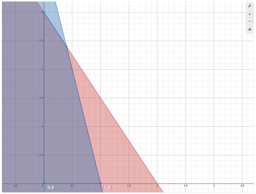
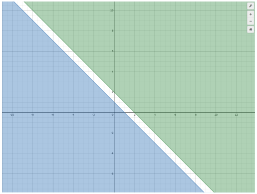
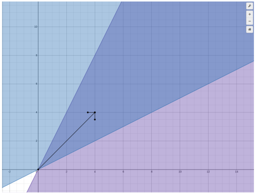
$$\begin{array}{ccccccc} x_2 & = & 240 & + & \frac{3}{5}x_4 & - & \frac{4}{5}x_3 \\ x_1 & = & 40 & - & \frac{2}{5}x_4 & - & \frac{1}{5}x_3 \\ \hline z & = & 640 & - & \frac{2}{5}x_4 & - & \frac{4}{5}x_3 \end{array}$$

Now $(x_1^*, x_2^*) = (40, 240)$. Increasing x_4 or x_3 will decrease z. THE END

Theorem (Fundamental Theorem of Linear Programming)

Let $P = \{x \mid Ax = b, x \geq 0\}$ *then either*

1. $P = \{\emptyset\}$
2. $P \neq \{\emptyset\}$ *and there exists a vertex* v *of* P *such that* $v \in \arg\min_{x \in P} c^\top x$
3. *There exists* $x, d \in \mathbb{R}^n$ *such that* $x + td \in P$ *for all* $t \geq 0$ *and* $\lim_{t \to \infty} c^\top (x + td) = \infty$.

Problem Notation

We will now formalize the definitions we introduced in the examples.

- There are n variables and m constraints
- The linear objective function $z = \sum_{j=1}^{n} c_j x_j$
- The m inequality constraints in standard form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \text{ for } i \in \{1, \ldots, m\}.$$

- The n positivity constraints $x_j \geq 0$, for $j \in \{1, \ldots, n\}$.
- x_i^* denotes the value of ith variable.
- We call $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a feasible solution if it satisfies the inequality and positivity constraints.

Dictionary Notation

- The slack variables $(x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^m$ (*variables d'écart*)
- The initial dictionary

$$
\begin{array}{lcl}
x_{n+1} & = & b_1 - \sum_{j=1}^{n} a_{1j} x_j \\
\vdots & & \\
x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j \\
\vdots & & \\
x_{n+m} & = & b_m - \sum_{j=1}^{n} a_{mj} x_j \\
\hline
z & = & \sum_{j=1}^{n} c_j x_j
\end{array}
$$

- Valid dictionary if m of the variables $(x_1, \ldots, x_{n+m})$ can be expressed as function of the remaining n variables.
- The m variables on the left-hand side are the basic variable (*variable de base*). The n variables on the right-hand side are the non-basic (*variable hors-base*).

Dictionary Notation

After row elimination operations we have a new basis.

- Basic variable set $I \subset \{1, \ldots, n+m\}$ and non-basic set $J = \{1, \ldots, n+m\} \setminus I$ with $|I| = m$ and $|J| = n$
- Current objective value $z^* = \sum_{j=1}^{n} c_j x_j^*$.
- For each basis set I there is a corresponding dictionary

$$\begin{array}{rcl} x_i & = & b_i' - \sum_{j \in J} a_{ij}' x_j, \text{ for } i \in I \\ \hline z & = & z^* + \sum_{j \in J} c_j' x_j, \end{array}$$

  where $a_{ij}', b_i', z^* \in \mathbb{R}$ are coefficients resulting from the row operations. For this to a feasible dictionary we require that $b_i' \geq 0$.
- A basic solution: $x_i^* = b_i'$ for $i \in I$ and $x_j^* = 0$ for $j \in J$.

Variable entering/leaving the basis

- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since

$$z = z^* + \sum_{j \in J} c'_j x_j.$$

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- How much can we increase x_{j_0}? Until there is a $x_i = 0$ since

$$x_i^* \; = \; b'_i - a'_{ij_0} x_{j_0}^* \; \geq \; 0$$

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- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since
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- If $a'_{ij_0} \leq 0$, then increasing $x_{j_0}^*$ will increase x_i^*
- If $a'_{ij} > 0$, then $x_{j_0}^* \leq b'_i / \ a'_{ij_0}$

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- Thus
$$x^*_{j_0} \ = \ \min_{i \in I, \ a'_{ij_0} > 0} \frac{b'_i}{a'_{ij_0}}$$

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- If $a'_{ij} > 0$, then $x^*_{j_0} \leq b'_i / a'_{ij_0}$
- Thus
$$x^*_{j_0} \;=\; \min_{i \in I,\, a'_{ij_0} > 0} \frac{b'_i}{a'_{ij_0}}$$
- In this case, which $x^*_i = 0$ (which i leaves the basis?)

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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if $c'_i \leq 0$ for all $i \in J$ **then**
    STOP;   # Optimal point found.

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Choose a variable j_0 to **enter the basis** from the set $j_0 \in \{j \in J \,:\, c'_j > 0\}$.
if $a'_{ij_0} \leq 0$ for all $i \in J$ **then**
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Choose a variable i_0 to **leave the basis** from the set $i_0 \in \arg \min\limits_{i \in I, a'_{ij_0} > 0} \left\{ \dfrac{b'_i}{a'_{ij_0}} \right\}$.

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$I \leftarrow (I \setminus \{i_0\})$ and $J \leftarrow J \cup \{i_0\}$ ▷ Move i_0 from basic to non-basic
for $i \in I$ **do**
  $a'_{i:} \leftarrow a'_{i:} - \frac{a'_{ij_0}}{a'_{i_0 j_0}} a'_{i_0:}$ ▷ Row elimination on pivot (i_0, j_0).

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$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:}$ and $a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$ ▷ Normalize the coefficient of $a'_{i_0 j_0}$

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$$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:} \quad \text{and} \quad a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$$ ▷ Normalize the coefficient of $a'_{i_0 j_0}$

$$c' \leftarrow c' - \frac{c'_{j_0}}{a'_{i_0 j_0}} a'_{i_0:}$$ ▷ Update the cost coefficients.

A Step of the Simplex Method

Input: $I = \{n+1, \dots, n+m\}$, $J = \{1, \dots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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▷ Row elimination on pivot (i_0, j_0).

$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:}$ and $a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$ ▷ Normalize the coefficient of $a'_{i_0 j_0}$

$$c' \leftarrow c' - \frac{c'_{j_0}}{a'_{i_0 j_0}} a'_{i_0:}$$

▷ Update the cost coefficients.

$I \leftarrow I \cup \{j_0\}$ and $J \leftarrow (J \setminus \{j_0\})$ ▷ Move j_0 from non-basic to basic

---

How to choose who enters the basis?

$$\boxed{j_0 \in \{j \in J \,:\, c'_j > 0\}}$$

1. The mad hatter rule: Choose the first one you see   costs: O(1)
2. Dantzig's 1st rule: $j_0 = \arg\max_{j \in J} c_j$ cost: O(n)
3. Dantzig's 2nd rule: Choose j_0 that maximizes the increase in z.

   $$j_0 = \arg\max_{j \in J} \left\{ c_j \min_{i \in I, a_{ij} > 0} \left\{ \frac{b_i}{a_{ij}} \right\} \right\} \quad costs : O(nm)$$

   Effective, but computationally expensive.
4. Bland's rule: Choose the smallest indices j_0 and i_0. That is, choose

   $$j_0 = \arg\min\{j \in J \,:\, c_j > 0\} \quad costs : O(n)$$

   $$i_0 = \min \left\{ \arg \min_{i \in I, a_{ij_0} > 0} \left\{ \frac{b_i}{a_{ij_0}} \right\} \right\}.$$

Degeneracy

Consider the problem

$$\begin{aligned} \max \quad & 2x_1 - x_2 + 8x_3 \\ & \qquad\qquad 2x_3 \le 1 \\ & 2x_1 - 4x_2 + 6x_3 \le 3 \\ & -x_1 + 3x_2 + 4x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Adding slack variables we have that

$$\begin{aligned} x_4 &= 1 + 0 + 0 - 2x_3 \\ x_5 &= 3 - 2x_1 + 4x_2 - 6x_3 \\ x_6 &= 2 + x_1 - 3x_2 - 4x_3 \\ \hline z &= 0 + 2x_1 - x_2 + 8x_3 \end{aligned}$$

Degeneracy

If any of the basic variables are zero, then we say that the solution is degenerate.

Consider the initial dictionary:

$$
\begin{array}{rl}
x_4 = & 1 + 0 + 0 - 2x_3 \\
x_5 = & 3 - 2x_1 + 4x_2 - 6x_3 \\
x_6 = & 2 + x_1 - 3x_2 - 4x_3 \\
\hline
z = & 0 + 2x_1 - x_2 + 8x_3
\end{array}
$$

If x_3 enters then who leaves?

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\hline
z &= 0 + 2x_1 - x_2 + 8x_3
\end{aligned}
$$

If x_3 enters then who leaves? Both x_5 and x_6 are set to zero, so either one. Choosing x_4 and pivoting on row 1 and column 4 we have.

$$
\begin{aligned}
x_3 &= 0.5 + 0 + 0 - 0.5x_4 \\
x_5 &= 0 - 2x_1 + 4x_2 + 3x_4 \\
x_6 &= 0 + x_1 - 3x_2 + 2x_4 \\
\hline
z &= 4 + 2x_1 - x_2 - 4x_4
\end{aligned}
$$

Only x_1 can enter the basis, but it doesn't increase in value :(

Full example in lecture notes.

Bland's rule for degeneracy

Bland's rule

Choose the smallest indices j_0 and i_0. That is, choose

$$j_0 = \arg\min\{j \in J \,:\, c_j > 0\}.$$

$$i_0 = \min\left\{\arg \min_{i \in I, a_{ij_0} > 0} \left\{\frac{b_i}{a_{ij_0}}\right\}\right\}.$$

Definition

A dictionary is degenerate if there are basic variables equal to zero.

Theorem

If Bland's rule is used on all degenerate dictionaries, then the simplex algorithm will not cycle.

Finding an initial feasible dictionary

$$
\begin{array}{lrrrl}
\max & x_1 & -x_2 & +x_3 & \\
 & 2x_1 & -x_2 & +2x_3 & \leq 4 \\
 & 2x_1 & -3x_2 & +x_3 & \leq -5 \\
 & -x_1 & +x_2 & -2x_3 & \leq -1 \\
 & x_1, & x_2, & x_3, & \geq 0.
\end{array}
$$

The point $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is not feasible.

Finding an initial feasible dictionary

$$\begin{array}{lllll} \max & x_1 & -x_2 & +x_3 & \\ & 2x_1 & -x_2 & +2x_3 & \leq 4 \\ & 2x_1 & -3x_2 & +x_3 & \leq -5 \\ & -x_1 & +x_2 & -2x_3 & \leq -1 \\ & x_1, & x_2, & x_3, & \geq 0. \end{array}$$

The point $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is not feasible.
Setup an auxiliary problem

$$\begin{array}{lllll} \max & -x_0 & & & \\ & 2x_1 & -x_2 & +2x_3 & -x_0 \leq 4 \\ & 2x_1 & -3x_2 & +x_3 & -x_0 \leq -5 \\ & -x_1 & +x_2 & -2x_3 & -x_0 \leq -1 \\ & x_1, & x_2, & x_3, & x_0 \geq 0. \end{array}$$

For x_0 big enough, it will be feasible. Setup initial dictionary

Initial phase one dictionary:

$$\begin{aligned}
x_4 &= 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\
x_5 &= -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\
x_6 &= -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\
w &= & & & & -x_0.
\end{aligned}$$

Pivot on "most infeasible" variable in the basis with the most negative value. Thus x_5 leaves the basis and x_0 enters the basis. Pivoting on row 2 and column 5:

Initial phase one dictionary:

$$\begin{aligned} x_4 &= 4 & -2x_1 & \quad +x_2 & -2x_3 & \quad +x_0 \\ x_5 &= -5 & -2x_1 & \quad +3x_2 & -x_3 & \quad +x_0 \\ x_6 &= -1 & +x_1 & \quad -x_2 & +2x_3 & \quad +x_0 \\ w &= & & & & \quad -x_0. \end{aligned}$$

Pivot on "most infeasible" variable in the basis with the most negative value. Thus x_5 leaves the basis and x_0 enters the basis. Pivoting on row 2 and column 5:

$r_1 \leftarrow r_1 - r_2$.

$r_3 \leftarrow r_3 - r_2$.

$w \leftarrow w + r_2$.

Initial phase one dictionary:

$$\begin{array}{rrrrr} x_4 = 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\ x_5 = -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\ x_6 = -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\ w = & & & & -x_0. \end{array}$$

Pivot on "most infeasible" variable in the basis with the most negative value. Thus x_5 leaves the basis and x_0 enters the basis. Pivoting on row 2 and column 5:

$r_1 \leftarrow r_1 - r_2$.
$r_3 \leftarrow r_3 - r_2$.
$w \leftarrow w + r_2$.

$$\begin{array}{rrrrr} x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\ x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\ x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\ w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5. \end{array}$$

Now x_2 enters and who leaves?

Initial phase one dictionary:

$$
\begin{array}{rrrrr}
x_4 = 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\
x_5 = -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\
x_6 = -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\
w = & & & & -x_0.
\end{array}
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$$
\begin{array}{rrrrr}
x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\
x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\
x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\
w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5.
\end{array}
$$

Now x_2 enters and who leaves? x_6 leaves the basis

$$
\begin{aligned}
x_4 &= 9 & +0 & & -2x_2 & & -x_3 & & +x_5 \\
x_0 &= 5 & 2x_1 & & -3x_2 & & +x_3 & & +x_5 \\
x_6 &= 4 & +3x_1 & & -4x_2 & & +3x_3 & & +x_5 \\
w &= -5 & -2x_1 & & +3x_2 & & -x_3 & & -x_5.
\end{aligned}
$$

Now x_2 enters and who leaves? x_6 leaves the basis. After pivoting

$$\begin{aligned}
x_4 &= 9 \quad +0 \quad -2x_2 \quad -x_3 \quad +x_5\\
x_0 &= 5 \quad 2x_1 \quad -3x_2 \quad +x_3 \quad +x_5\\
x_6 &= 4 \quad +3x_1 \quad -4x_2 \quad +3x_3 \quad +x_5\\
w &= -5 \quad -2x_1 \quad +3x_2 \quad -x_3 \quad -x_5.
\end{aligned}$$

Now x_2 enters and who leaves? x_6 leaves the basis. After pivoting

$$\begin{aligned}
x_2 &= 1 \quad +0.75x_1 \quad +0.75x_3 \quad +0.25x_5 \quad -0.25x_6\\
x_0 &= 2 \quad -0.25x_1 \quad -1.25x_3 \quad +0.25x_5 \quad +0.75x_6\\
x_4 &= 7 \quad -1.5x_1 \quad -2.5x_3 \quad +0.5x_5 \quad +0.5x_6\\
w &= -2 \quad +0.25x_1 \quad +1.25x_3 \quad -0.25x_5 \quad -0.75x_6.
\end{aligned}$$

Who enters the basis now?

$$\begin{array}{rrrrr} x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\ x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\ x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\ w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5. \end{array}$$

Now x_2 enters and who leaves? x_6 leaves the basis. After pivoting

$$\begin{array}{rrrrr} x_2 = 1 & +0.75x_1 & +0.75x_3 & +0.25x_5 & -0.25x_6 \\ x_0 = 2 & -0.25x_1 & -1.25x_3 & +0.25x_5 & +0.75x_6 \\ x_4 = 7 & -1.5x_1 & -2.5x_3 & +0.5x_5 & +0.5x_6 \\ w = -2 & +0.25x_1 & +1.25x_3 & -0.25x_5 & -0.75x_6. \end{array}$$

Who enters the basis now? x_3

Who leaves the basis?

$$\begin{array}{rcccc} x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\ x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\ x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\ w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5. \end{array}$$

Now x_2 enters and who leaves? x_6 leaves the basis. After pivoting

$$\begin{array}{rcccc} x_2 = 1 & +0.75x_1 & +0.75x_3 & +0.25x_5 & -0.25x_6 \\ x_0 = 2 & -0.25x_1 & -1.25x_3 & +0.25x_5 & +0.75x_6 \\ x_4 = 7 & -1.5x_1 & -2.5x_3 & +0.5x_5 & +0.5x_6 \\ w = -2 & +0.25x_1 & +1.25x_3 & -0.25x_5 & -0.75x_6. \end{array}$$

Who enters the basis now? x_3
Who leaves the basis?

$$x_0 \geq 0 \quad \Rightarrow \quad 2 - 1.25x_3 \geq 0 \quad \Rightarrow \quad x_3 \geq 2/1.25 = 1.6$$

$$x_4 \geq 0 \quad \Rightarrow \quad 7 - 2.5x_3 \geq 0 \quad \Rightarrow \quad x_3 \geq 7/2.5 = 2.8$$

x_0 leaves the basis!

$$\begin{array}{rrrrr} x_2 = 1 & +0.75x_1 & +0.75x_3 & +0.25x_5 & -0.25x_6 \\ x_0 = 2 & -0.25x_1 & -1.25x_3 & +0.25x_5 & +0.75x_6 \\ x_4 = 7 & -1.5x_1 & -2.5x_3 & +0.5x_5 & +0.5x_6 \\ \hline w = -2 & +0.25x_1 & +1.25x_3 & -0.25x_5 & -0.75x_6. \end{array}$$

Pivoting on row 2 and column 3:

$r_1 \leftarrow r_1 + \frac{0.75}{1.25} r_2 = r_1 + 0.6 r_2.$

$r_3 \leftarrow r_3 - 2r_2.$

$w \leftarrow w + r_2.$

$$\begin{array}{rrrrr} x_2 = 2.2 & +0.6x_1 & +0.4x_5 & +0.2x_6 & -0.6x_0 \\ x_3 = 1.6 & -0.2x_1 & +0.2x_5 & +0.6x_6 & -0.8x_0 \\ x_4 = 3 & -x_1 & & -x_6 & +2x_0 \\ \hline w = & & & & . \end{array}$$

Feasible basis without x_0!

$$
\begin{array}{rrrrr}
x_2 = 1 & +0.75x_1 & +0.75x_3 & +0.25x_5 & -0.25x_6 \\
x_0 = 2 & -0.25x_1 & -1.25x_3 & +0.25x_5 & +0.75x_6 \\
x_4 = 7 & -1.5x_1 & -2.5x_3 & +0.5x_5 & +0.5x_6 \\
\hline
w = -2 & +0.25x_1 & +1.25x_3 & -0.25x_5 & -0.75x_6.
\end{array}
$$

Pivoting on row 2 and column 3:

$r_1 \leftarrow r_1 + \frac{0.75}{1.25} r_2 = r_1 + 0.6 r_2.$

$r_3 \leftarrow r_3 - 2 r_2.$

$w \leftarrow w + r_2.$

$$
\begin{array}{rrrrr}
x_2 = 2.2 & +0.6x_1 & +0.4x_5 & +0.2x_6 & -0.6x_0 \\
x_3 = 1.6 & -0.2x_1 & +0.2x_5 & +0.6x_6 & -0.8x_0 \\
x_4 = 3 & -x_1 & & -x_6 & +2x_0 \\
\hline
w = & & & & .
\end{array}
$$

Feasible basis without x_0! Remove column with x_0 and replace w with z.

$$
\begin{array}{rrrr}
x_2 = 2.2 & +0.6x_1 & +0.4x_5 & +0.2x_6 \\
x_3 = 1.6 & -0.2x_1 & +0.2x_5 & +0.6x_6 \\
x_4 = 3 & -x_1 & & -x_6 \\
\hline
z = & +x_1 & -x_2 & x_3
\end{array}
$$

$$\begin{aligned}
x_2 &= 2.2 \quad +0.6x_1 \quad +0.4x_5 \quad +0.2x_6 \\
x_3 &= 1.6 \quad -0.2x_1 \quad +0.2x_5 \quad +0.6x_6 \\
x_4 &= 3 \quad\;\; -x_1 \qquad\qquad\qquad\; -x_6
\end{aligned}$$

Eliminate base variables x_2 and x_3 from z:

$$\begin{aligned} x_2 &= 2.2 \quad +0.6x_1 \quad +0.4x_5 \quad +0.2x_6 \\ x_3 &= 1.6 \quad -0.2x_1 \quad +0.2x_5 \quad +0.6x_6 \\ x_4 &= 3 \quad -x_1 \quad -x_6 \end{aligned}$$

Eliminate base variables x_2 and x_3 from z:

$$z = x_1 - x_2 + x_3$$

$$\begin{aligned} x_2 &= 2.2 \quad +0.6x_1 \quad +0.4x_5 \quad +0.2x_6 \\ x_3 &= 1.6 \quad -0.2x_1 \quad +0.2x_5 \quad +0.6x_6 \\ x_4 &= 3 \qquad -x_1 \qquad\qquad\qquad -x_6 \end{aligned}$$

Eliminate base variables x_2 and x_3 from z:

$$\begin{aligned} z &= x_1 - x_2 + x_3 \\ &= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\ &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6. \end{aligned}$$

$$\begin{array}{rllll} x_2 = & 2.2 & +0.6x_1 & +0.4x_5 & +0.2x_6 \\ x_3 = & 1.6 & -0.2x_1 & +0.2x_5 & +0.6x_6 \\ x_4 = & 3 & -x_1 & & -x_6 \end{array}$$

Eliminate base variables x_2 and x_3 from z:

$$\begin{aligned} z &= x_1 - x_2 + x_3 \\ &= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\ &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6. \end{aligned}$$

So the initial basis is

$$\begin{array}{rllll} x_2 = & 2.2 & +0.6x_1 & +0.4x_5 & +0.2x_6 \\ x_3 = & 1.6 & -0.2x_1 & +0.2x_5 & +0.6x_6 \\ x_4 = & 3 & -x_1 & & -x_6 \\ \hline z = & -0.6 & +0.2x_1 & -0.2x_3 & +0.4x_6 \end{array}$$

Now apply the simplex again!

Upper Bounds Using Duality

The LP in standard form

$$\max_{x} z \overset{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \leq b,$$
$$x \geq 0, \tag{LP}$$

We want to find $w \in \mathbb{R}$ so that $z = c^\top x \leq w$ for all $x \in \mathbb{R}^n$.
Combine rows of constraints?

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$$x \ge 0, \qquad \text{(LP)}$$

We want to find $w \in \mathbb{R}$ so that $z = c^\top x \le w$ for all $x \in \mathbb{R}^n$.
Combine rows of constraints?
Look for $y \ge 0 \in \mathbb{R}^m$ so that $y^\top A \ge c^\top$ so that

$$c^\top x \le (y^\top A)x \le y^\top b =: w.$$

Can we make this upper bound as tight as possible? Yes, by minimizing $y^\top b$. That is, we need to the *dual* linear program.

Dual definition

$$\max_x z \stackrel{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \leq b,$$
$$x \geq 0, \qquad (P) \text{ Primal} \qquad (1)$$

$$\min_y w \stackrel{\text{def}}{=} y^\top b$$
$$\text{subject to } A^\top y \geq c,$$
$$y \geq 0. \qquad (D) \text{ Dual} \qquad (2)$$

Dual definition

$$\begin{aligned} \max_x \ & z \stackrel{\text{def}}{=} c^\top x \\ \text{subject to } & Ax \le b, \\ & x \ge 0, \end{aligned} \qquad (P) \text{ Primal} \qquad (1)$$

$$\begin{aligned} \min_y \ & w \stackrel{\text{def}}{=} y^\top b \\ \text{subject to } & A^\top y \ge c, \\ & y \ge 0. \end{aligned} \qquad (D) \text{ Dual} \qquad (2)$$

Exe: Show that the dual of the dual is the primal.

Dual definition

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$$\begin{aligned} \min_y \; & w \stackrel{\text{def}}{=} y^\top b \\ \text{subject to } & A^\top y \geq c, \\ & y \geq 0. \end{aligned} \qquad (D) \text{ Dual} \tag{2}$$

Exe: Show that the dual of the dual is the primal.

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ is a feasible point for (1) and $y \in \mathbb{R}^m$ is a feasible point for (2) then

$$c^\top x \leq y^\top Ax \leq y^\top b. \tag{3}$$

Weak Duality

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ is a feasible point for (1) and $y \in \mathbb{R}^m$ is a feasible point for (2) then

$$c^\top x \leq y^\top Ax \leq y^\top b. \tag{4}$$

Consequently

- *If (1) has an unbounded solution, that is $c^\top x \to \infty$, then*

Weak Duality

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ *is a feasible point for* (1) *and* $y \in \mathbb{R}^m$ *is a feasible point for* (2) *then*

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Consequently

- *If* (1) *has an unbounded solution, that is* $c^\top x \to \infty$, *then there exists no feasible point* y *for* (2)
- *If* (2) *has an unbounded solution, that is* $y^\top b \to -\infty$, *then*

Weak Duality

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ is a feasible point for (1) *and $y \in \mathbb{R}^m$ is a feasible point for* (2) *then*

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Consequently

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- *If* (2) *has an unbounded solution, that is $y^\top b \to -\infty$, then there exists no feasible point x for* (1)
- *If x and y are primal and dual feasible, respectively, and $c^\top x = y^\top b$, then*

Weak Duality

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If $x \in \mathbb{R}^n$ is a feasible point for (1) and $y \in \mathbb{R}^m$ is a feasible point for (2) then

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Consequently

- *If (1) has an unbounded solution, that is $c^\top x \to \infty$, then there exists no feasible point y for (2)*
- *If (2) has an unbounded solution, that is $y^\top b \to -\infty$, then there exists no feasible point x for (1)*
- *If x and y are primal and dual feasible, respectively, and $c^\top x = y^\top b$, then x and y are the primal and dual optimal points, respectively.*

Strong Duality

Theorem (Strong Duality)

If (1) *or* (2) *is feasible, then* $z^* = w^*$. *Moreover, if* c^* *is the cost vector of the optimal dictionary of the primal problem* (1), *that is, if*

$$z = z^* + \sum_{i=1}^{n+m} c_i^* x_i, \tag{5}$$

then $y_i^* = -c_{n+i}^*$ *for* $i = 1, \ldots, m$.

Thus distance to optimal is given by

$$z - w \;=\; y^\top b - c^\top x \;\geq\; 0.$$

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Thus distance to optimal is given by

$$z - w \ = \ y^\top b - c^\top x \ \geq \ 0.$$

Proof: First $c_i^* \leq 0$ for $i = 1, \dots, m + n$ because dictionary is optimal. Consequently $y_i^* = -c_{n+i}^* \geq 0$ for $i = 1, \dots, m$.

Strong duality: Proof I

By the definition of the slack variables we have that

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad \text{for } i = 1, \dots, m. \tag{6}$$

Consequently, setting $y_i^* = -c_{n+i}^*$, we have that

$$\begin{aligned}
z &\overset{(5)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=n+1}^{n+m} c_i^* x_i \\
&\overset{(6)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
&= z^* - \sum_{i=1}^{m} y_i^* b_i + \sum_{j=1}^{n} \left(c_j^* + \sum_{i=1}^{m} y_i^* a_{ij} \right) x_j \\
&\overset{\text{def of } z}{=} \sum_{j=1}^{n} c_j x_j, \qquad \forall x_1, \dots, x_n.
\end{aligned} \tag{7}$$

Last line followed by definition $z = \sum_{j=1}^{n} c_j x_j$. Since the above holds for all $x \in \mathbb{R}^n$, we can match the coefficients.

Strong duality: Proof II

$$z^* - \sum_{i=1}^{m} y_i^* b_i + \sum_{j=1}^{n} \left(c_j^* + \sum_{i=1}^{m} y_i^* a_{ij} \right) x_j \quad = \quad \sum_{j=1}^{n} c_j x_j .$$

Matching coefficients on x_j's we have

$$z^* \quad = \quad \sum_{i=1}^{m} y_i^* b_i \tag{8}$$

$$c_j \quad = \quad c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}, \qquad \text{for } j = 1, \ldots, n. \tag{9}$$

Since $c_j^* \leq 0$ for $j = 1, \ldots, n$, the above is equivalent to

$$z^* \quad = \quad \sum_{i=1}^{m} y_i^* b_i \tag{10}$$

$$\sum_{i=1}^{m} y_i^* a_{ij} \quad \leq \quad c_j, \qquad \text{for } j = 1, \ldots, n. \tag{11}$$

(11) $\Rightarrow$ y_i^* is feasible for (2). (10) $\Rightarrow$ $z^* = \sum_{i=1}^{m} y_i^* b_i = w$, consequently by weak duality the y_i^*'s are dual optimal. $\square$

How to calculate dual solution y?

By strong duality

$$c^\top x^* \quad = \quad (y^*)^\top A x^* \quad = \quad (y^*)^\top b.$$

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Subtracting $(y^*)^\top A x^*$ from all sides of the above gives

$$\big(\underbrace{c - A^\top y^*}_{\geq 0}\big)^\top x^* \quad = \quad 0 \quad = \quad (y^*)^\top \big(\underbrace{b - A x^*}_{\geq 0}\big).$$

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$$\big(\underbrace{c - A^\top y^*}_{\geq 0}\big)^\top x^* \quad = \quad 0 \quad = \quad (y^*)^\top \big(\underbrace{b - Ax^*}_{\geq 0}\big).$$

Re-writing the above in element form we have that

$$\sum_{j=1}^{n} \big(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\big) x_j^* \quad = \quad 0 \quad = \quad \sum_{i=1}^{m} y_i^* \big(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\big).$$

How to calculate dual solution y?

By strong duality

$$c^\top x^* \quad = \quad (y^*)^\top A x^* \quad = \quad (y^*)^\top b.$$

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$$\big(\underbrace{c - A^\top y^*}_{\geq 0}\big)^\top x^* \quad = \quad 0 \quad = \quad (y^*)^\top\big(\underbrace{b - Ax^*}_{\geq 0}\big).$$

Re-writing the above in element form we have that

$$\sum_{j=1}^{n}\big(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\big) x_j^* \quad = \quad 0 \quad = \quad \sum_{i=1}^{m} y_i^* \big(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\big).$$

Sum over positive numbers equal zero thus

$$y_i^*\big(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\big) \quad = \quad 0, \quad \forall i = 1, \ldots, m.$$

$$x_j^*\big(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\big) \quad = \quad 0, \quad \forall j = 1, \ldots, n.$$

How to calculate dual solution y?

$$
\begin{aligned}
y_i^* \big(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\big) &= 0, \quad \forall i = 1, \ldots, m. \\
x_j^* \big(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\big) &= 0, \quad \forall j = 1, \ldots, n.
\end{aligned}
$$

This gives the following rule for computing y^*.

$$
\begin{aligned}
\sum_{i=1}^{n} a_{ij} y_i^* &= c_j, \quad \forall j \in \{1, \ldots, n\}, \;\; x_j^* > 0. \\
y_i^* &= 0, \quad \forall i \in \{1, \ldots, m\}, \;\; b_i > \sum_{j=1}^{n} a_{ij} x_j^*.
\end{aligned}
$$

Question: If x^* is non-degenerate, how many $x_i^* > 0$?

Complementary slackness

Since $b_i > \sum_{j=1}^{n} a_{ij}x_j^* \;\Rightarrow\; x_{n+i}^* > 0$ we have

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Finally

$$\sum_{i=1}^{n} a_{ij} y_i^* \;=\; c_j \quad \Rightarrow\; A_J^\top y^* \;=\; c_J \qquad (J \text{ indices of Basic variables})$$

Exercise on calculating dual variables

$$\begin{array}{llll} \max z = & 4x_2 & +3x_2 & \\ & 5x_1 & +3x_2 & \leq 30 \\ & 2x_1 & +3x_2 & \leq 24 \\ & x_1 & +3x_2 & \leq 18 \end{array}$$

If $x_1^* = 3, x_2^* = 5$

Then $y_1^* = \frac{3}{4}, y_2^* = 0, y_3^* = \frac{1}{4}$

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$$5x_1^* + 3x_2^* = 5 * 3 + 3 * 5 = 30 \quad \Rightarrow \quad y_1^* \neq 0$$

$$2x_1^* + 3x_2^* = 2 * 3 + 3 * 5 = 21 \; < \; 24 \quad \Rightarrow \quad y_2^* = 0$$

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Exercise on calculating dual variables

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G.,R & P Richtárik, Randomized Iterative Methods for Linear Systems arXiv:1506.03296